Tripoli university
Faculty of engineering
EE department EE313
Electrical polarization tutorial

Problem # 1

A thin rod of cross section A along the x-axis extends from $x=0$ to $x=L$. The polarization of the road is along its length and is given by $P_x = ax^2+b$. Calculate ρ_p and ρ_{sp} at each end. Show that the total bound charge vanishes in this case.

Solution

$$\rho_p = -\vec{\nabla}\cdot\vec{P} = -\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right) = -2ax$$

We can sketch the rod in fig(1)

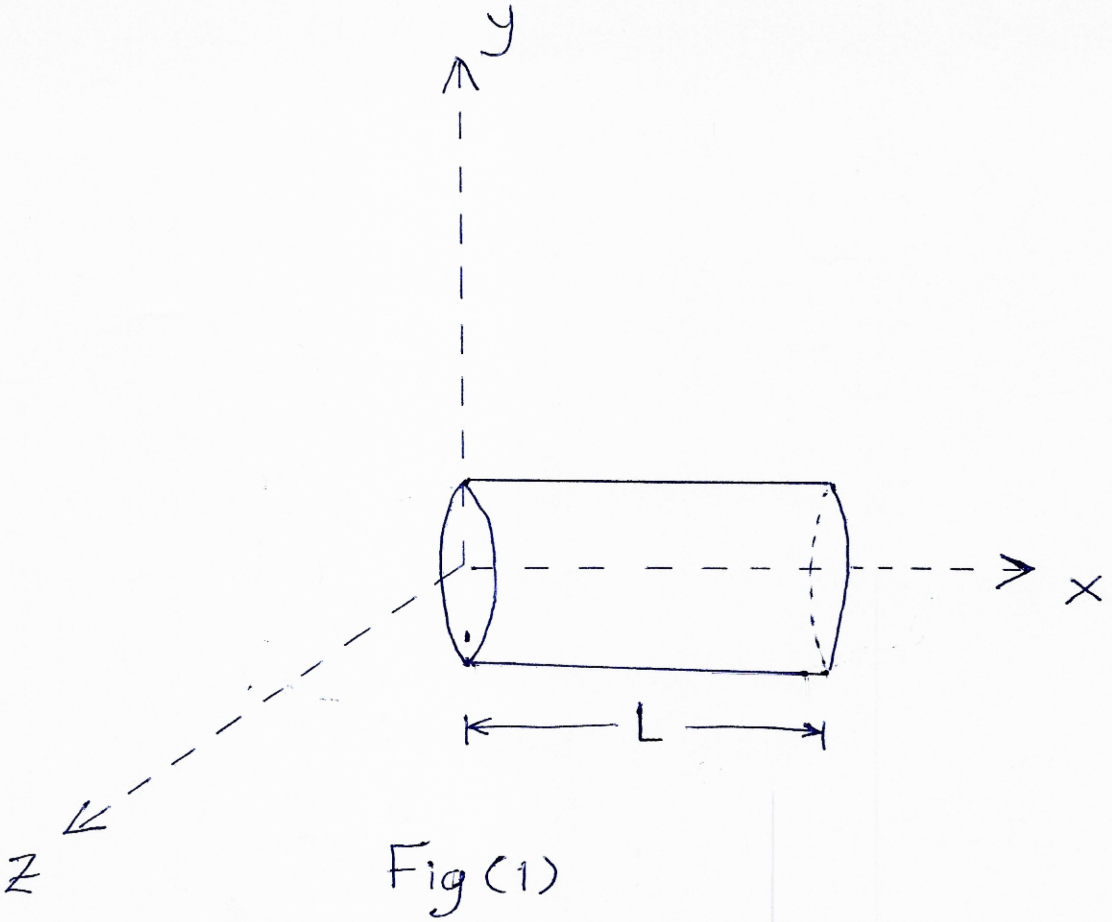

Fig (1)

At first end of the rod $(x=0)$:

$\rho_p = -2a(0) = 0$

At the other end of the rod $(x=L)$:

$\rho_p = -2aL$

From boundary conditions of the polarization vector when region1 is free space (see equation 3-47):

$\vec{n} \cdot \vec{P}_2 = \rho_{sp}$

where $\vec{n}$ is a unit vector normal to the surface separating the two regions and pointing into region1.

$\vec{n}$ in our problem is $-\vec{a}_x$ at $x=0$ and $\vec{n}$ is $\vec{a}_x$ at $x=L$.

at surface $x=0$:-

$$\rho_{sp} = -(ax^2+b)\Big|_{x=0} = -b$$

at surface $x=L$:-

$$\rho_{sp} = (ax^2+b)\Big|_{x=L} = aL^2+b$$

Now, we'll prove that the total charge on the rod must be zero:

$$Q_t = \int_v \rho_p \, dv + \underset{\text{at } x=a}{\int_s \rho_{sp} \, ds} + \underset{\text{at } x=L}{\int_s \rho_{sp} \, ds}$$

$$= \int_0^L \rho_p A \, dx + \underset{\text{at } x=0}{\rho_{sp} A} + \underset{\text{at } x=L}{\rho_{sp} A}$$

$$= -A(2a)\int_0^L x \, dx - bA + aL^2A + bA$$

$$= -2Aa\left[\frac{x^2}{2}\right]_0^L + aL^2A$$

$$= aL^2A - aL^2A = 0$$

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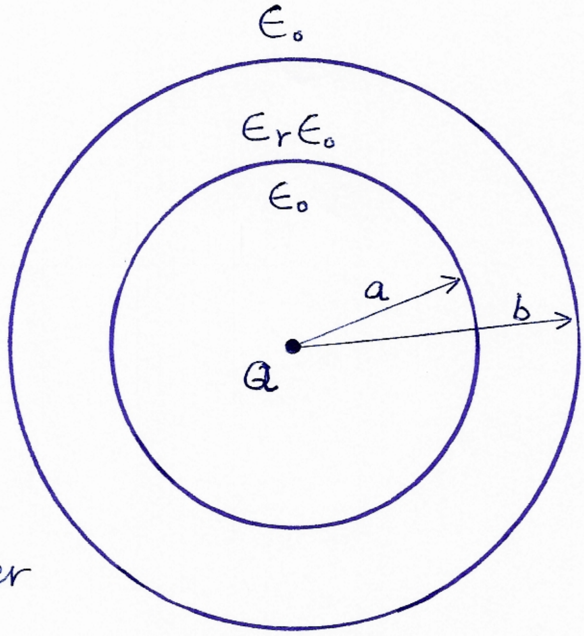
Problem #2



Consider the fig. as a spherical dielectric shell so that $\epsilon = \epsilon_o \epsilon_r$ for $a<r<b$ and $\epsilon = \epsilon_o$ for $0<r<a$. If a charge Q is placed at the center of the shell, find :-

a) $\vec{P}$ for $a<r<b$. b) ρ_p for $a<r<b$. c) ρ_{sp} at $r=a, r=b$.

Solution

Applying Gauss law for materials (equation 3-37):

$$\oint_S \vec{D}\cdot d\vec{s} = \int_V \rho_v \, dv$$

where the surface to be chosen is a sphere with Q at its center and radius r.
where $\int_V \rho_v \, dv$ is simply the charge enclosed by the surface which is the point charge Q.

Hence, :-

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} D_r\, r^2 \sin\theta\, d\theta\, d\phi = Q$$

$$4\pi r^2 D_r = Q$$

$$\therefore \vec{D} = \vec{a}_r \frac{Q}{4\pi r^2}$$

which is valid for any region.

Now, $\vec{E}$ in the dielectric is from equation (3-30b):-

$$\vec{E} = \frac{\vec{D}}{\epsilon_r \epsilon_o} = \vec{a}_r \frac{Q}{4\pi \epsilon_r \epsilon_o r^2}$$

And $\vec{P}$ is found within the dielectric using equation (3-23):-

$$\vec{P} = \vec{D} - \epsilon_o \vec{E} = \vec{a}_r \left[\frac{Q}{4\pi r^2} - \frac{Q}{4\pi \epsilon_r r^2} \right]$$

$$= \vec{a}_r \frac{Q}{4\pi r^2} \left(1 - \frac{1}{\epsilon_r}\right)$$

b) $$\rho_p = -\vec{\nabla} \cdot \vec{P}$$

$$= -\left[\frac{1}{r^2} \frac{\partial}{\partial r}(r P_r) + 0 + 0 \right]$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{Q}{4\pi}\left(1-\frac{1}{\epsilon_r}\right)\right) = 0$$

c)

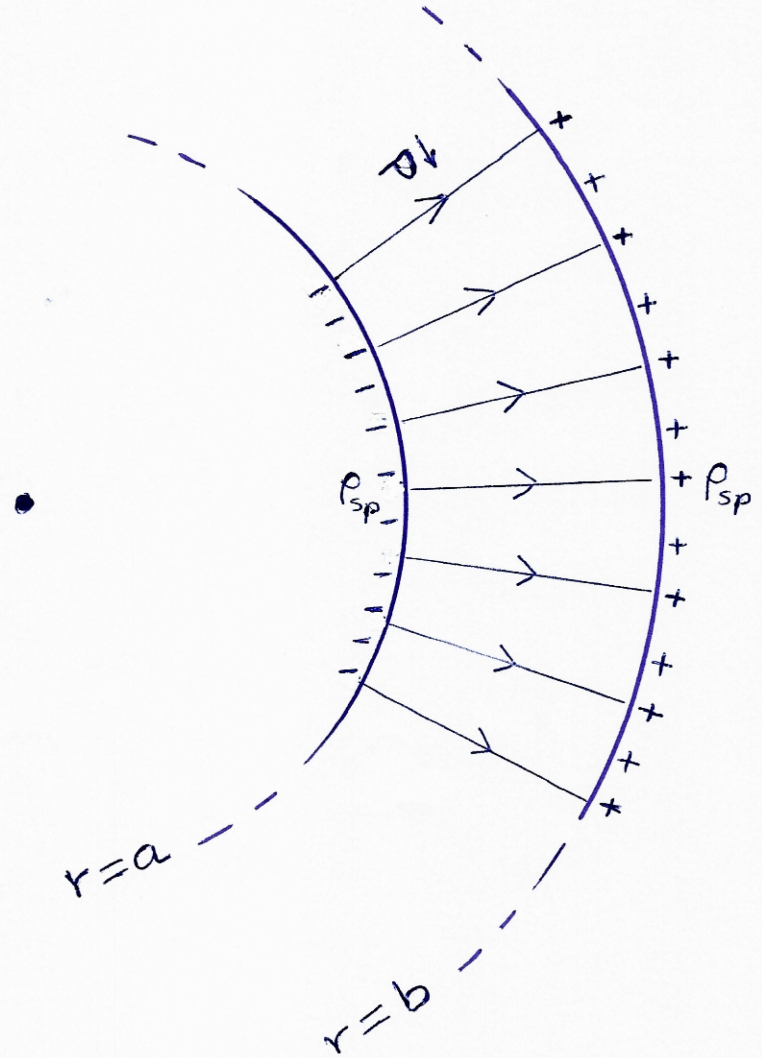


Fig (2)

For boundary $r=a$:-

for region 1 (free space) and region 2 (dielectric)

$\vec{n} = -\vec{a_r}$.

$$\rho_{sp} = \vec{n}\cdot\vec{P_2} = -\left[\frac{Q}{4\pi r^2}\left(1-\frac{1}{\epsilon_r}\right)\right]_{r=a} = -\frac{Q}{4\pi a^2}\left(1-\frac{1}{\epsilon_r}\right)$$

For boundary $r=b$:-

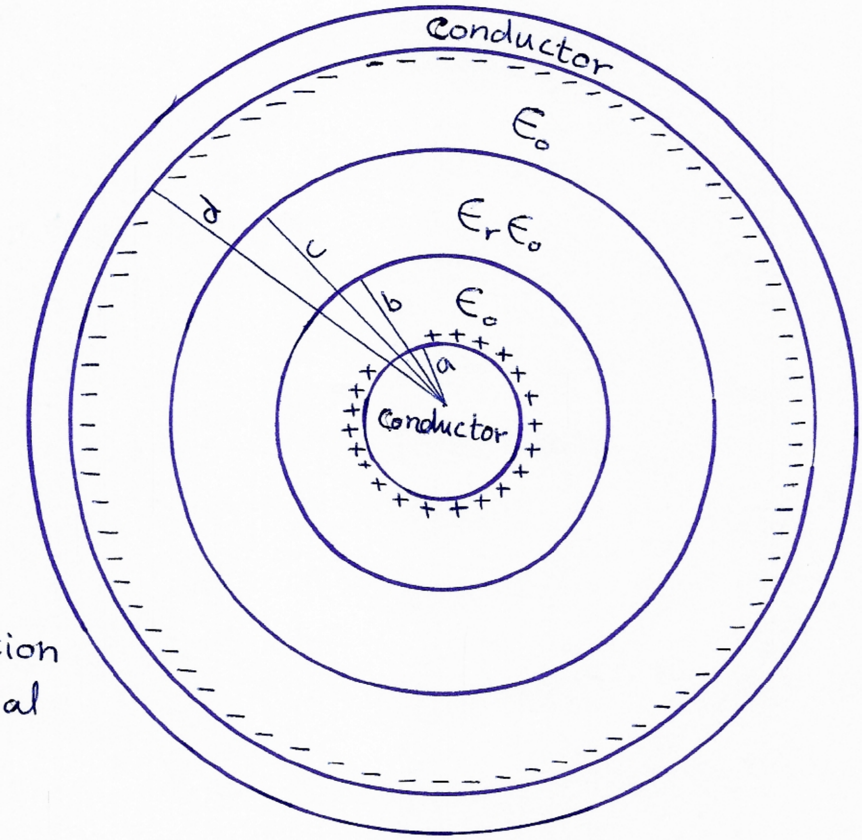
For region(1) (free space) and region(2) (dielectric)
$\vec{n} = \vec{a_r}$. from equation 3-47 :-

$$\rho_{sp} = \vec{n} \cdot \vec{P_2} = \left[\frac{Q}{4\pi r^2}\left(1-\frac{1}{\epsilon_r}\right)\right]_{r=b} = \frac{Q}{4\pi b^2}\left(1-\frac{1}{\epsilon_r}\right).$$

Problem#3

This problem is (3-11) in Johnk.

Solution



Fig(3)

Cross section of the coaxial pair.

If we apply Gauss law (3-37) by choosing the surface as a cylinder with radius ρ and $a < \rho < d$:-

$$\int_{z=0}^{l} \int_{\phi=0}^{2\pi} D_\rho \, \rho \, d\phi \, dz = Q$$

where the total charge on the inner conductor for length l is given in the question to be Q.

$$2\pi \rho l D_\rho = Q$$

$$\therefore \vec{D} = \vec{a}_\rho \frac{Q}{2\pi \rho l}$$

which is valid in dielectric region as well as in free-space regions.

b)

In the dielectric region ($b < \rho < c$) :-

$$\vec{E} = \frac{\vec{D}}{\epsilon_r \epsilon_0} = \vec{a}_\rho \frac{Q}{2\pi \epsilon_r \epsilon_0 \rho l}$$

and the polarization $\vec{P}$:-

$$\vec{P} = \vec{D} - \epsilon_o \vec{E}$$

$$= \vec{a}_\rho \left[\frac{Q}{2\pi \rho l} - \frac{Q}{2\pi \epsilon_r \rho l} \right]$$

$$= \vec{a}_\rho \frac{Q}{2\pi \rho l} \left(1 - \frac{1}{\epsilon_r}\right)$$

$$\cancel{\rho_p = \vec{\nabla} \cdot \vec{P} = \frac{1}{\rho}(\rho P_\rho)}$$

$$\rho_p = -\vec{\nabla} \cdot \vec{P} = -\left[\frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho P_\rho) + 0 + 0 \right]$$

$$= -\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\frac{Q}{2\pi l} \left(1 - \frac{1}{\epsilon_r}\right) \right) = 0$$

c)

For the boundary $\rho = a$ (see fig(3)) we have conductor and free space. If we apply the boundary conditions of the D vectors by choosing the conductor to be region(2) and the free space to be region(1) then $\vec{n} = \vec{a}_\rho$. From equation (3-45):-

$$\rho_s = \vec{n} \cdot \vec{D}_1 = \left[\frac{Q}{2\pi \rho l} \right]_{\rho = a} = \frac{Q}{2\pi a l}$$

At boundary $\rho = d$, let the conductor to be region(2) and the free space to be region(1) again where $\vec{n} = -\vec{a}_\rho$:-

$$\rho_s = \vec{n} \cdot \vec{D} = -\left[\frac{Q}{2\pi \rho l}\right]_{\rho = d} = -\frac{Q}{2\pi d l}$$

For the boundary $\rho = b$, let region (1) to be the free space and region(2) to be the dielectric. By applying the boundary conditions of the $\vec{P}$ vectors (equation 3-47) where $\vec{n} = -\vec{a}_\rho$:

$$\rho_{sp} = \vec{n} \cdot \vec{P}_2 = -\left[\frac{Q}{2\pi \rho l}\left(1 - \frac{1}{\epsilon_r}\right)\right]_{\rho = b} = -\frac{Q}{2\pi b l}\left(1 - \frac{1}{\epsilon_r}\right)$$

For the boundary $\rho = c$, let region (1) to be the free space and region(2) to be the dielectric ($\vec{n} = \vec{a}_\rho$):

$$\rho_{sp} = \vec{n} \cdot \vec{P}_2 = \left[\frac{Q}{2\pi \rho l}\left(1 - \frac{1}{\epsilon_r}\right)\right]_{\rho = c} = \frac{Q}{2\pi c l}\left(1 - \frac{1}{\epsilon_r}\right).$$

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